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# DYNAMIC DEFORMATION OF AN ELASTOVISCOPLASTIC HOLLOW SPHERE

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*The stress–strain state of a hollow sphere under time-dependent loading is studied using the constitutive relations for a hardening compressible elastoviscoplastic sold. Analytical solutions are obtained for displacement fields in the elastic and plastic regions. Time dependences of the reciprocal of the radius of the elastoplastic boundary are constructed, and the effect of the physicomechanical parameters on the radius of the elastoplastic boundary is determined.*

**Key words:** *plasticity, viscosity, hardening, strain, loading function, compressibility, dilatancy.*

Ershov [1] investigated the axisymmetric instability of a thick-walled spherical shell under uniform pressure in a static formulation using small elastoplastic deformation theory. Sporykhin [2] studied the problem of complex media using flow theory under the assumption of incompressibility of the material. Semykina [3] solved the problem of triaxial extension of elastoplastic space with a spherical cavity under periodic loading in a dynamic formulation. In the present paper, we consider the dynamic deformation of a hollow compressible hardening elastoviscoplastic sphere of radius $R$ with inner radius $a$ (Fig. 1). The outer surface of the sphere is subjected to distributed load $P$, and the contour of the internal cavity to load $p$; these loads can be expressed as

$$P = P^0 + \sum_{k=1}^{n} P^k \,\mathrm{e}^{\omega_k t+\gamma_k}, \qquad p = p^0 + \sum_{k=1}^{n} p^k \,\mathrm{e}^{\omega_k t+\gamma_k}\,. \tag{1}$$

Here $k = 0, \ldots, n$, $0 \leqslant t \leqslant \infty$, $\omega_k$ and $\gamma_k$ are known constants ($\omega_k < 0$).

The problem is solved in spherical coordinates using dimensionless variables. The quantities having the dimension of length are normalized to the radius of the elastoplastic boundary $r_s$, and the quantities having the dimension of stress are normalized to the shear modulus $\mu$.

In view of axial symmetry, the constitutive equations of the problem have the following form:
for the equations of motion,

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}\,(\sigma_r - \sigma_\theta) - \frac{\rho_0}{r}\,\frac{\partial^2 u}{\partial t^2} = 0 \tag{2}$$

($\rho_0$ is the dimensionless density of the material; $u = u_r$);
for the Cauchy relations,

$$\varepsilon_r = \frac{du}{dr}, \qquad \varepsilon_\theta = \frac{u}{r}; \tag{3}$$

for Hooke's law for the stress in the elastic region,

$$\sigma_r = 2\varepsilon_r + \lambda_0(\varepsilon_r + 2\varepsilon_\theta), \qquad \sigma_\theta = 2\varepsilon_\theta + \lambda_0(\varepsilon_r + 2\varepsilon_\theta), \qquad \lambda_0 = \lambda/\mu. \tag{4}$$

For a compressible hardening elastoviscoplastic solid in the Ivlev–Sporykhin model [2–5], the loading function is written as

$$F = \alpha\sigma_1 - (S_{ij} - c_0 e^p_{ij} - \eta_0 \dot e^p_{ij})(S_{ij} - c_0 e^p_{ij} - \eta_0 \dot e^p_{ij}) - K_0 = 0,$$

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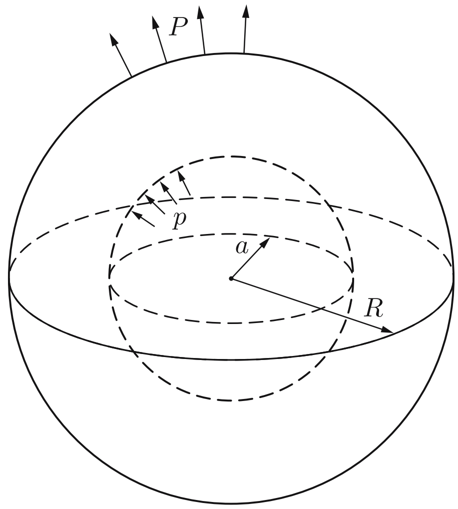


Fig. 1. Hollow sphere under external and internal dynamic loads.

where $\alpha$ is the dilatancy rate, $\sigma_1 = \sigma_{kk}/3$ is the first invariant of the stress tensor, $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ and $e_{ij} = \varepsilon_{ij} - \varepsilon_{kk}\delta_{ij}/3$ are the stress and strain tensor deviators, respectively, $K_0$ is the yield point, $c_0$ and $\eta_0$ are the hardening and viscosity parameters. The relations for the total strains in the plastic region are written as $\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}$, and the associated plastic flow law, according to [2, 4, 5], is written as

$$\frac{\partial \varepsilon^p_r}{\partial t} = \zeta\Big(\frac{\alpha}{3} + \frac{S_r - c_0 e^p_r - \eta_0 \dot{e}^p_r}{K_0 - \alpha\sigma_1}\Big) + \Psi(\sigma_1)\dot{\sigma}_1,$$

$$\frac{\partial \varepsilon^p_\theta}{\partial t} = \zeta\Big(\frac{\alpha}{3} + \frac{S_\theta - c_0 e^p_\theta - \eta_0 \dot{e}^p_\theta}{K_0 - \alpha\sigma_1}\Big) + \Psi(\sigma_1)\dot{\sigma}_1$$

($\Psi = \text{const}$). The boundary conditions and conjugation conditions become

$$\sigma_r\Big|_{r=a_*} = \frac{p}{\mu}, \quad \sigma_r\Big|_{r=qa_*} = \frac{P}{\mu}, \quad [\sigma_r]_{r=1} = 0, \quad [u]_{r=1} = 0, \quad a_* = \frac{a}{r_s}, \quad q = \frac{R}{a}. \tag{5}$$

Under the assumption that, at the onset of plastic flow $t = t_*$, the plastic region begins to form from the boundaries of the internal cavity of the sphere, the initial conditions are specified as

$$a_*\Big|_{t=t_*} = 1. \tag{6}$$

According to (1), the required functions become

$$\Phi(r,t) = \Phi^0(r) + \sum_{k=1}^{n} \Phi^k(r)\,\mathrm{e}^{\omega_k t + \gamma_k}\,. \tag{7}$$

Using the equations of motion (2), Cauchy relations (3), and Hooke's law (4), we obtain the following system of $n+1$ differential equations for displacements in the elastic region:

$$\frac{d^2u^k}{dr^2} + \frac{2}{r}\frac{du^k}{dr} - \Big(2 + \frac{\rho_0\omega_k^2}{2+\lambda_0}\,\beta_k\Big)\frac{u^k}{r^2} = 0, \quad k = 0,\dots,N, \qquad \beta_k = \begin{cases} 0, & k = 0, \\ 1, & k > 0, \end{cases}$$

from which we have

$$u^e = a^0_{11} r + \frac{a^0_{12}}{r} + \sum_{k=1}^{n} (a^k_{11} r^{n^k_1} + a^k_{12} r^{n^k_2})\,\mathrm{e}^{\omega_k t + \gamma_k}, \tag{8}$$

where $n^k_{1,2} = 1/2 \pm \sqrt{9/4 + \rho_0\omega_k^2/(2+\lambda_0)}$ $(k = 1,\dots,n)$ and $a^k_{11}$ are integration constants.

According to Hooke's law, the stress components in the elastic region become

$$\sigma_r^e = (2+3\lambda_0)a_{11}^0 + \frac{\lambda_0-2}{r^2}\,a_{12}^0 + \sum_{k=1}^{n}\Big[(2\lambda_0+n_1^k(2+\lambda_0))a_{11}^k r^{n_1^k-1} + (2\lambda_0+n_2^k(2+\lambda_0))a_{12}^k r^{n_2^k-1}\Big]\,\mathrm{e}^{\omega_k t+\gamma_k},$$
$$\sigma_\theta^e = (2+3\lambda_0)a_{11}^0 + \frac{\lambda_0+2}{r^2}\,a_{12}^0 + \sum_{k=1}^{n}\Big[(2+\lambda_0(2+n_1^k))a_{11}^k r^{n_1^k-1} + (2+\lambda_0(2+n_2^k))a_{11}^k r^{n_2^k-2}\Big]\,\mathrm{e}^{\omega_k t+\gamma_k}. \tag{9}$$

Due to axial symmetry, we have

$$S_\theta = S_\varphi = -S_r/2, \qquad e_\theta = e_\varphi = -e_r/2.$$

The loading function and the associated plastic flow law are written as

$$\alpha\sigma_1 + \sqrt{3/2}\,(S_r - c_0 e_r^p - \eta_0 \dot{e}_r^p) - K_0 = 0, \qquad \dot{\varepsilon}_r^p = \xi(\alpha/3 + \sqrt{2/3}\,) + \Psi\dot{\sigma}_1, \qquad \dot{\varepsilon}_\theta^p = \xi(\alpha/3 - \sqrt{1/6}\,) + \Psi\dot{\sigma}_1.$$

In view of relations (1), we have

$$\alpha\sigma_1^k + \sqrt{3/2}\,(S_r^k - [c_0 + \eta_0\omega_k\beta_k]e_r^{p^k}) - K_0\beta_k = 0,$$
$$\omega_k(\varepsilon_r^{p^k} - \Psi\sigma_1^k)\beta_k = \xi^k(\alpha/3 + \sqrt{2/3}\,), \qquad \omega_k(\varepsilon_\theta^{p^k} - \Psi\sigma_1^k)\beta_k = \xi^k(\alpha/3 - \sqrt{1/6}\,),$$

from which, taking into account Hooke's law (4) and Cauchy relations (3), we obtain

$$\sigma_r^p = \sum_{k=0}^{n} \frac{1}{\Delta^k}\Big(\Delta_{11}^k\,\frac{du^k}{dr} + \Delta_{12}^k\,\frac{u^k}{r} + \Delta_{13}^k K_0\beta_k\Big)\,\mathrm{e}^{\omega_k t+\gamma_k},$$
$$\sigma_\theta^p = \sum_{k=0}^{n} \frac{1}{\Delta^k}\Big(\Delta_{21}^k\,\frac{du^k}{dr} + \Delta_{22}^k\,\frac{u^k}{r} + \Delta_{23}^k K_0\beta_k\Big)\,\mathrm{e}^{\omega_k t+\gamma_k}, \tag{10}$$

where

$$\Delta^k = 3\alpha\lambda_0\sqrt{6} + 6\alpha^2(2+3\lambda_0) + 3(2+c_0+\beta_k\eta_0\omega_k)[6\Psi(2+3\lambda_0)+3\lambda_0+6],$$
$$\Delta_{11}^k = 4(2+3\lambda_0)(\sqrt{2}\alpha-\sqrt{3})^2 + 12(c_0+\beta_k\eta_0\omega_k)[2\Psi(2+3\lambda_0)+3\lambda_0+3],$$
$$\Delta_{12}^k = 4(2+3\lambda_0)(6-2\alpha^2-\sqrt{6}\alpha) - 6(c_0+\beta_k\eta_0\omega_k)[4\Psi(2+3\lambda_0)-3\lambda_0],$$
$$\Delta_{13}^k = (2+3\lambda_0)(6\alpha+12\sqrt{6}\Psi) + 3\sqrt{6}\,(4+3\lambda_0),$$
$$\Delta_{21}^k = 2(2+3\lambda_0)(6+2\alpha^2-\sqrt{6}\alpha) + 6(c_0+\beta_k\eta_0\omega_k)[3\lambda_0-2\Psi(2+3\lambda_0)],$$
$$\Delta_{22}^k = 4(2+3\lambda_0)(\alpha+\sqrt{6})^2 + 12(c_0+\beta_k\eta_0\omega_k)[\Psi(2+3\lambda_0)+3\lambda_0+3],$$
$$\Delta_{23}^k = 6(2+3\lambda_0)(\alpha-\sqrt{6}\,\Psi) - 6\sqrt{6}.$$

Substitution of relations (10) into the equations of motion (2) yields the following system of equations for the displacement component in the plastic region:

$$b_0^k\,\frac{d^2u^k}{dr^2} + b_1^k\,\frac{du^k}{dr} - b_2^k\,\frac{u^k}{r^2} + b_3^k\,\frac{K_0}{r} = 0, \qquad k = 0,\ldots,n. \tag{11}$$

Here

$$b_0^k = 4(2+3\lambda_0)(\sqrt{2}\alpha-\sqrt{3}\,)^2 + 12(c_0+\beta_k\eta_0\omega_k)[2\Psi(2+3\lambda_0)+3\lambda_0+3],$$
$$b_1^k = 4(2+3\lambda_0)(\alpha-\sqrt{6}\,)^2 + 6(c_0+\beta_k\eta_0\omega_k)[8\Psi(2+3\lambda_0)+9\lambda_0+12],$$
$$b_2^k = 4(2+3\lambda_0)(4\alpha^2+5\sqrt{6}\,\alpha+6) + (c_0+\beta_k\eta_0\omega_k)[48\Psi(2+3\lambda_0)+18(4+3\lambda_0)]$$
$$+\,\rho_0\omega_k^2\beta_k[6(2+3\lambda_0)\alpha^2 + 3\lambda_0\alpha\sqrt{6} + 9(2+c_0+\beta_k\eta_0\omega_k)(2\Psi[2+3\lambda_0]+\lambda_0+2)],$$
$$b_3^k = 18\beta_k\sqrt{6}\,[\lambda_0+2-2(2+3\lambda_0)\Psi].$$

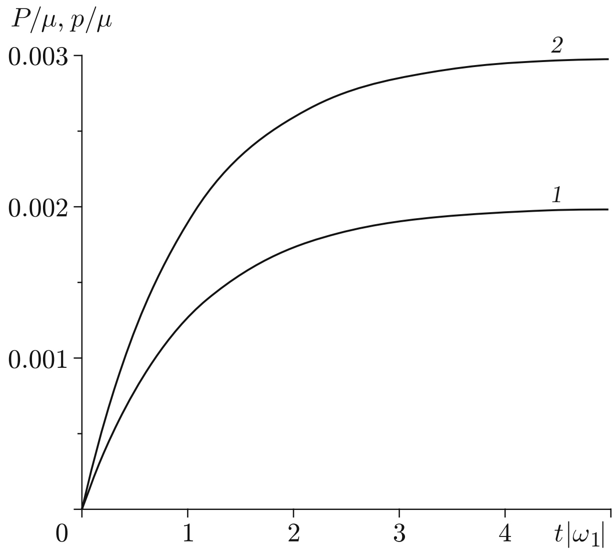


Fig. 2

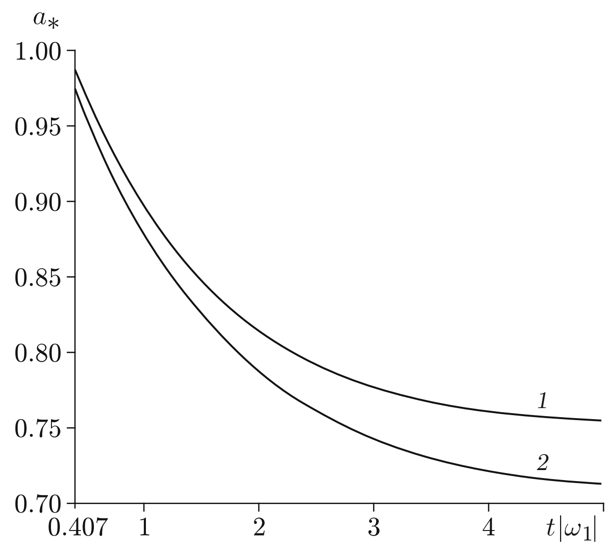


Fig. 3

Fig. 2. External $P/\mu$ (1) and internal $p/\mu$ (2) loads versus time ($\omega_k = -k \cdot 10^{-2}$ and $\gamma_k = 0$, where $k = 1$, 2, and 3).

Fig. 3. Reciprocal of the radius of the elastoplastic boundary versus time for $\alpha = 0.1$, $c_0 = 0.1$, and viscosity $\eta_0 = 0.05$ (1) and 0.10 (2).

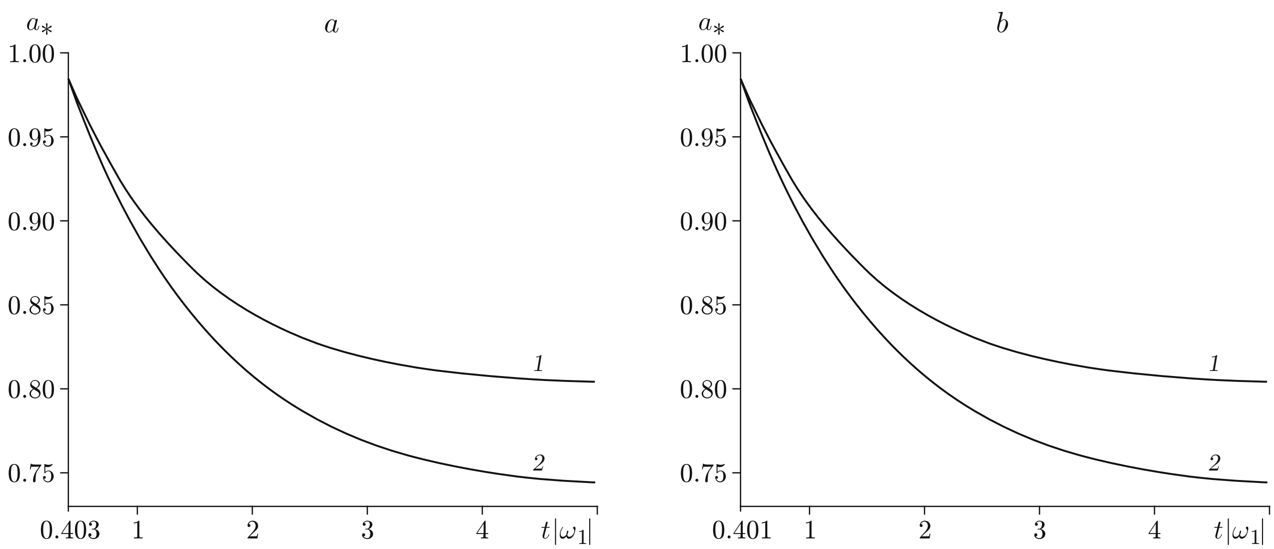


Fig. 4. Reciprocal of the radius of the elastoplastic boundary versus time for $\eta_0 = 0.05$ and dilatancy rate $\alpha = 0.2$ (a) and, 0.3 (b) and hardening parameter $c_0 = 0.10$ (curve 1) and 0.05 (curve 2).

We write the solution of system (11) as

$$u^p = \sum_{k=0}^{n} (a_{21}^k r^{m_1^k} + a_{22}^k r^{m_2^k})\, e^{\omega_k t+\gamma_k} + \frac{b_3^0 K_0}{b_2^0 - b_1^0}\, r + a_{21}^0 r^{m_1^0} + a_{22}^0 r^{m_2^0},$$

where

$$m_{1,2}^k = \Big(b_1^k - b_0^k \pm \sqrt{(b_1^k - b_0^k)^2 + 4b_0^k b_2^k}\Big)/(2b_0^k), \tag{12}$$

and $a_{ij}^k$ are integration constants.

According to (10)–(12), the stress components in the plastic region are written as

$$\begin{aligned}
\sigma_r^p &= \sum_{k=1}^{n} \frac{1}{\Delta^k} \Big[(\Delta_{11}^k m_1^k + \Delta_{12}^k) a_{21}^k r^{m_1^k - 1} + (\Delta_{11}^k m_2^k + \Delta_{12}^k) a_{22}^k r^{m_2^k - 1}\Big]\, e^{\omega_k t+\gamma_k} + \frac{1}{\Delta^0}\Big[\Delta_{13}^0 + \frac{b_3^0(\Delta_{11}^0 + \Delta_{12}^0)}{b_2^0 - b_1^0}\Big] K_0,\\
\sigma_\theta^p &= \sum_{k=1}^{n} \frac{1}{\Delta^k} \Big[(\Delta_{21}^k m_1^k + \Delta_{22}^k) a_{21}^k r^{m_1^k - 1} + (\Delta_{21}^k m_2^k + \Delta_{22}^k) a_{22}^k r^{m_2^k - 1}\Big]\, e^{\omega_k t+\gamma_k} + \frac{1}{\Delta^0}\Big[\Delta_{23}^0 + \frac{b_3^0(\Delta_{21}^0 + \Delta_{22}^0)}{b_2^0 - b_1^0}\Big] K_0.
\end{aligned} \tag{13}$$

From the obtained solutions (8), (9), (12), and (13) subject to appropriate boundary conditions, conjugation conditions (5), and initial condition (6), we finally express the integration constants as

$$a_{ij}^k = g_{ij}^k / g^k, \qquad i = 1, 2, \quad j = 1, 2, \quad k = 1, \ldots, n,$$

$$\begin{aligned}
g^k &= d_{11}^k w_1^k [a_*^{m_1^k - 1} d_{21}^k (d_{12}^k - d_{22}^k) + d_{22}^k a_*^{m_2^k - 1} (d_{21}^k - d_{12}^k)]\\
&\quad - d_{12}^k w_2^k [a_*^{m_1^k - 1} d_{21}^k (d_{11}^k - d_{22}^k) + d_{22}^k a_*^{m_2^k - 1} (d_{21}^k - d_{12}^k)],
\end{aligned}$$

$$\begin{aligned}
g_1^k &= P_0^k [a_*^{m_1^k - 1} d_{21}^k (d_{12}^k - d_{22}^k) + d_{22}^k a_*^{m_2^k - 1} (d_{21}^k - d_{12}^k)] - d_{12}^k w_2^k [(p_0^k - f_1[1-\beta_k])(d_{21}^k - d_{22}^k)\\
&\quad + (d_{21}^k a_*^{m_1^k - 1} (f_1 - f_2 d_{22}^k) + d_{22}^k a_*^{m_2^k - 1} [f_2 d_{21}^k - f_1])(1-\beta_k)],
\end{aligned}$$

$$\begin{aligned}
g_2^k &= d_{21}^k w_1^k [(p_0^k - f_1[1-\beta_k])(d_{21}^k - d_{22}^k) + (a_*^{m_1^k - 1} d_{21}^k [f_1 - f_2 d_{22}^k] + a_*^{m_2^k - 1} d_{22}^k [f_2 d_{21}^k - f_1])(1-\beta_k)]\\
&\quad - P_0^k [d_{21}^k a_*^{m_1^k - 1} (d_{11}^k - d_{22}^k) + d_{22}^k a_*^{m_2^k - 1} (d_{21}^k - d_{11}^k)],
\end{aligned}$$

$$\begin{aligned}
g_3^k &= (p_0^k - f_1[1-\beta_k])[d_{11}^k w_1^k (d_{22}^k - d_{12}^k) + d_{12}^k w_2^k (d_{11}^k - d_{12}^k)]\\
&\quad - d_{22}^k a_*^{m_2^k - 1} [P_0^k (d_{11}^k - d_{12}^k) + (d_{12}^k w_2^k [f_1 - f_2 d_{11}^k] + d_{11}^k [f_2 d_{12}^k - f_1])(1-\beta_k)],
\end{aligned}$$

$$\begin{aligned}
g_4^k &= d_{21}^k a_*^{m_1^k - 1} [P_0^k (d_{11}^k - d_{12}^k) + (d_{12}^k w_2^k [f_1 - f_2 d_{11}^k] + d_{11}^k [f_2 d_{12}^k - f_1])(1-\beta_k)]\\
&\quad - (p_0^k - f_1[1-\beta_k])[d_{11}^k w_1^k (d_{21}^k - d_{12}^k) + d_{12}^k w_2^k (d_{11}^k - d_{21}^k)],
\end{aligned}$$

$$w_1^k = \begin{cases} 1, & k = 0,\\ (q a_*)^{n_1^k - 1}, & k > 0, \end{cases} \qquad w_2^k = \begin{cases} (q a_*)^{-2}, & k = 0,\\ (q a_*)^{n_2^k - 1}, & k > 0, \end{cases}$$

$$d_{11}^k = (2 + \lambda_0)(1 + n_1^k [1-\beta_k]), \qquad d_{12}^k = (2 + \lambda_0)(1 + n_2^k [1-\beta_k]),$$

$$d_{21}^k = (\Delta_{11}^k m_1^k + \Delta_{12}^k)/\Delta^k, \qquad d_{22}^k = (\Delta_{11}^k m_2^k + \Delta_{12}^k)/\Delta^k,$$

$$f_1 = \frac{1}{\Delta^0}\Big[\Delta_{13}^k + \frac{b_3^0(\Delta_{11}^0 + \Delta_{12}^0)}{b_2^0 - b_1^0}\Big] K_0, \qquad f_2 = \frac{b_3^0}{b_2^0 - b_1^0}\, K_0.$$

The radius of the elastoplastic boundary is found from the equation

$$\sum_{k=1}^{n} \frac{1}{g^k}\Big[(\Delta_{21}^k m_1^k + \Delta_{22}^k)\,\frac{g_{21}^k}{\Delta^k} + (\Delta_{21}^k m_2^k + \Delta_{22}^k)\,\frac{g_{22}^k}{\Delta^k}$$

$$- (2+\lambda_0(2+n_1^k))g_{11}^k - (2+\lambda_0(2+n_2^k))g_{12}^k\Big]\,\mathrm{e}^{\omega_k t+\gamma_k}$$

$$= \frac{1}{g^0}\,\Big[(2+3\lambda_0)g_{11}^0 + (\lambda_0+2)g_{12}^0\Big] - \frac{1}{\Delta^0}\Big[\Delta_{23}^0 + \frac{b_3^0(\Delta_{21}^0+\Delta_{22}^0)}{b_2^0 - b_1^0}\Big]K_0.$$

Numerical calculations were performed for the case of time-dependent external and internal dynamic loads for $\Psi = K_0 = 0.01$, $\rho_0 = 0.001$, and $\lambda_0 = 1.5$ (Fig. 2). The results of the numerical experiment are presented in Figs. 3 and 4, which show the dependence of the reciprocal $a_*$ of the radius of the elastoplastic boundary on time $t$ for various values of the physicomechanical parameters [4].

It should be noted that, for $P = P^0$ and $p = p^0$, the obtained solutions coincide with the solutions of the static problem given in [2].

Figures 3 and 4 show the effect of the viscosity and hardening parameter on the behavior of the elastoplastic boundary. It is evident that the plastic deformation region increases with increasing viscosity and decreases with increasing hardening parameter. Thus, with time, the radius of the elastoplastic boundary tends to a constant value.

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